Unique Fixed Point Theorems for Generalized Hybrid Mappings in Hilbert Spaces and Applications (Nonlinear Analysis and Convex Analysis)

AUTHOR(S):

高橋, 渉

数理解析研究所講究録
第 1963 巻 2015 年 162-169

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慶応義塾大学自然科学研究教育センター, 台湾国立中山大学応用数学系
高橋渉 (Wataru Takahashi)
Keio Research and Education Center for Natural Sciences, Keio University, Japan and
Department of Applied Mathematics, National Sun Yat-sen University, Taiwan

Abstract. In this article, we prove unique fixed point theorems for symmetric generalized hybrid mappings and symmetric more generalized hybrid mappings in Hilbert spaces. Using these results, we prove unique fixed point theorems for strict pseudo-contractions in Hilbert spaces. In particular, we obtain an extension of the famous strong convergence theorem with implicit iteration which was proved by Browder in 1967.

1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping $U : C \to H$ is called a *widely strict pseudo-contraction* [20] if there exists $r \in \mathbb{R}$ with $r < 1$ such that

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + r\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C.$$

We call such U a *widely r-strict pseudo-contraction.* If $0 \leq r < 1$, then U is a *strict pseudo-contraction* [4]. Furthermore, if $r = 0$, then U is *nonexpansive.* In 1967, Browder [3] proved the famous strong convergence theorem with implicit iteration in a Hilbert space.

Theorem 1.1 ([3]). *Let H be a Hilbert space, let C be a bounded closed convex subset of H and let T be a nonexpansive mapping of C into C. Let $u \in C$ and define a sequence $\{y_{\alpha_n}\}$ in C by*

$$y_{\alpha_n} = \alpha_n u + (1 - \alpha_n)Ty_{\alpha_n}, \quad \forall \alpha_n \in (0, 1).$$

Then $\{y_{\alpha_n}\}$ converges strongly to Pu as $\alpha_n \to 0$, where P is the metric projection of H onto $F(T)$.

If we replace a nonexpansive mapping T in Theorem 1.1 by a strict pseudo-contraction, does such a theorem hold?

Kawasaki and Takahashi [8] defined the following class of nonlinear mappings in a Hilbert space which covers contractive mappings and generalized hybrid mappings in the sense of Kocourek, Takahashi and Yao [9]. A mapping T from C into H is said to be *widely generalized* *hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0, \quad \forall x, y \in C.$$

Motivated by Kawasaki and Takahashi [8], Takahashi, Wong and Yao [21] introduced a more broad class of nonlinear mappings than the class of widely generalized hybrid mappings in a Hilbert space. A mapping $T : C \to H$ is said to be *symmetric generalized hybrid* [21] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + \beta(\|x - Ty\|^2 + \|Tx - y\|^2) + \gamma\|x - y\|^2 \\ + \delta(\|x - Tx\|^2 + \|y - Ty\|^2) \leq 0, \quad \forall x, y \in C.$$

Such a mapping T is also called *$(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid.* They proved the following fixed point theorem for symmetric generalized hybrid mappings in a Hilbert space.

Theorem 1.2 ([21]). *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta > 0$ *and* (3) $\delta \geq 0$ *hold. Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, \ldots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition* (1).

Furthermore, they introduced the following class of nonlinear mappings which contains the class of symmetric generalized hybrid mappings. A mapping T from C into H is called *symmetric more generalized hybrid* [21] if there exist $\alpha, \beta, \gamma, \delta, \zeta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + \beta(\|x - Ty\|^2 + \|Tx - y\|^2) + \gamma\|x - y\|^2 \\ + \delta(\|x - Tx\|^2 + \|y - Ty\|^2) + \zeta\|x - y - (Tx - Ty)\|^2 \leq 0, \quad \forall x, y \in C.$$

Such a mapping T is also called *$(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid.* They also proved the following fixed point theorem.

Theorem 1.3 ([21]). *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ *and* (3) $\delta + \zeta \geq 0$ *hold. Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, \ldots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition* (1).

In the case when the mappings in Theorems 1.2 and 1.3 have unique fixed points, what kind of iterations can we use to find such unique fixed points? This question is natural.

In this article, motivated by Theorems 1.2 and 1.3, we prove unique fixed point theorems for symmetric generalized hybrid mappings and symmetric more generalized hybrid mappings in Hilbert spaces. Using these results, we prove unique fixed point theorems for strict pseudo-contractions in Hilbert spaces. In particular, we obtain an extension of the famous strong convergence theorem with implicit iteration which was proved by Browder [3].

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see [16]. Furthermore, in a Hilbert space, we have that

$$2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2$$

for all $x, y, z, w \in H$. Let C be a nonempty subset of H and let T be a mapping from C into H. We denote by $F(T)$ the set of fixed points of T. A mapping T from C into H with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|Tx - u\| \leq \|x - u\|$ for any $x \in C$ and $u \in F(T)$. A nonexpansive mapping with a fixed point is quasi-nonexpansive. It is well-known that if $T : C \to H$ is quasi-nonexpansive and C is closed and convex, then $F(T)$ is closed and convex; see Itoh and Takahashi [7]. It is not difficult to prove such a result in a Hilbert space. Let D be a nonempty closed convex subset of H and $x \in H$. We know that there exists a unique nearest point $z \in D$ such that $\|x - z\| = \inf_{y \in D} \|x - y\|$. We denote such a correspondence by $z = P_D x$. The mapping P_D is called the *metric projection* of H onto D. It is known that P_D is nonexpansive and

$$\langle x - P_D x, P_D x - u\rangle \geq 0$$

for all $x \in H$ and $u \in D$; see [16] for more details.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ or $\mu_n x_n$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [15] for the proof of existence of a Banach limit and its other elementary properties. Using means and the Riesz theorem, we can obtain the following result; see [13], [14] and [15].

Lemma 2.1. *Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n : n \in \mathbb{N}\}$ such that*

$$\mu_n\langle x_n, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H.$$

Let H be a Hilbert space and let C be a nonempty closed convex subset of H. A mapping $U : C \to H$ is called *extended hybrid* [6] if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned}
\alpha(1+\gamma)\|Ux - Uy\|^2 &+ (1 - \alpha(1+\gamma))\|x - Uy\|^2 \\
&\leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\
&\quad - (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2, \quad \forall x, y \in C.
\end{aligned}$$

Such a mapping U is called $(\alpha,\ \beta,\ \gamma)$-*extended hybrid*. We know the following fixed point result for strict pseudo-contractions in a Hilbert space.

Lemma 2.2 ([19]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $0 \le k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction. Then, U is a (1,0,-k)-extended hybrid mapping and $F(U)$ is closed and convex. If, in addition, C is bounded and U is of C into itself, then $F(U)$ is nonempty.*

The following lemma was proved by Takahashi, Wong and Yao [20].

Lemma 2.3 ([20]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let A, U and T be mappings of C into H such that $U = I - A$ and $T = 2\alpha\, U + (1 - 2\alpha)I$. Then, the following are equivalent:*

(a) A is an α-inverse-strongly monotone mapping, i.e.,

$$\alpha\|Ax - Ay\|^2 \le \langle x - y, Ax - Ay\rangle, \quad \forall x, y \in C;$$

(b) U is a widely $(1 - 2\alpha)$-strict pseudo-contraction, i.e.,

$$\|Ux - Uy\|^2 \le \|x - y\|^2 + (1 - 2\alpha)\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C;$$

(c) U is a $(1, 0, 2\alpha - 1)$-extended hybrid mapping, i.e.,

$$\begin{aligned} &2\alpha\|Ux - Uy\|^2 + (1 - 2\alpha)\|x - Uy\|^2 \\ &\quad \le (2\alpha - 1)\|Ux - y\|^2 + 2(1 - \alpha)\|x - y\|^2 \\ &\qquad - (2\alpha - 1)\|x - Ux\|^2 - (2\alpha - 1)\|y - Uy\|^2, \quad \forall x, y \in C; \end{aligned}$$

(d) T is a nonexpansive mapping.

Using Lemma 2.3, we obtain the following result.

Lemma 2.4 ([20]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $k < 1$ and let A, U and T be mappings of C into H such that $U = I - A$ and $T = (1 - k)U + kI$. Then, the following are equivalent:*

(a) A is a $\frac{1-k}{2}$-inverse-strongly monotone mapping;
(b) U is a widely k-strict pseudo-contraction;
(c) U is a $(1, 0, -k)$-extended hybrid mapping;
(d) T is a nonexpansive mapping.

The following lemma was also proved by Takahashi, Wong and Yao [19].

Lemma 2.5 ([19]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let α, β, γ be real numbers and let $U : C \to H$ be an (α, β, γ)-extended hybrid mapping with $1 + \gamma > 0$. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Using Lemmas 2.2 and 2.5, we have the following result obtained by Marino and Xu [12].

Lemma 2.6 ([12]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $0 \le k < 1$ and $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

3 Unique fixed point theorems

We first prove the following unique fixed point theorem for symmetric generalized hybrid mappings in a HIlbert space whose domains are not bounded.

Theorem 3.1 ([18]). *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma > 0$, (2) $\beta \leq 0$, (3) $\beta + \gamma \leq 0$, *and* (4) $\beta + \delta \geq 0$ *hold. Then*

(i) T has a unique fixed point u in C;
(ii) for every $z \in C$, the sequence $\{T^n z\}$ converges to u.

Using Theorem 3.1, we prove the following fixed point theorem.

Theorem 3.2. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma > 0$, (2) $\beta \leq \zeta$, (3) $\beta + \gamma \leq 0$, *and* (4) $\beta + \delta \geq 0$ *hold. Then*

(i) T has a unique fixed point u in C;
(ii) for every $z \in C$, the sequence $\{T^n z\}$ converges to u.

The following is an extension of Theorem 3.2.

Theorem 3.3. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself which satisfies the conditions* (1) $\alpha + 2\beta + \gamma > 0$, (2) *there exists* $\lambda \in [0, 1)$ *such that* $(\alpha + \beta)\lambda + \zeta - \beta \geq 0$, (3) $\beta + \gamma \leq 0$ *and* (4) $\beta + \delta \geq 0$. *Then*

(i) T has a unique fixed point u in C;
(ii) for every $z \in C$, the sequence $\{(\lambda I + (1 - \lambda T)^n z\}$ converges to u.

Next, we obtain a unique fixed point theorem for symmetric generalized hybrid mappings in a Hilbert space whose domains are bounded.

Theorem 3.4 ([18]). *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma > 0$, (2) $\alpha + \beta + \delta > 0$ *and* (3) $\delta \geq 0$ *hold. Then*

(i) T has a unique fixed point u in C;
(ii) for every $z \in C$, a subsequence $\{T^{n_i} z\}$ of $\{T^n z\}$ converges to u.

In particular, if $\beta + \gamma \leq 0$, then $\{T^n z\}$ for all $z \in C$ converges to u.

Using Theorem 3.4, we prove the following fixed point theorem.

Theorem 3.5. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma > 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ *and* (3) $\delta + \zeta \geq 0$ *hold. Then*

(i) T has a unique fixed point u in C;
(ii) for every $z \in C$, a subsequence $\{T^{n_i} z\}$ of $\{T^n z\}$ converges to u.

In particular, if $\beta + \gamma \le 0$, then $\{T^n z\}$ for all $z \in C$ converges to u.

The following theorem is an extension of Theorem 3.5.

Theorem 3.6. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself which satisfies the conditions (1) $\alpha + 2\beta + \gamma > 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ and (3) there exists $\lambda \in [0, 1)$ such that $(\alpha + \beta)\lambda + \delta + \zeta \ge 0$. Then*

(i) T has a unique fixed point u in C;
(ii) for every $z \in C$, a subsequence $\{(\lambda I + (1 - \lambda)T)^{n_i} z\}$ of $\{(\lambda I + (1 - \lambda)T)^n z\}$ converges to u.

In particular, if $\beta + \gamma \le 0$, then $\{(\lambda I + (1 - \lambda)T)^n z\}$ for all $z \in C$ converges to u.

4 Applications

Using Theorem 3.1, we can first prove the following fixed point theorem.

Theorem 4.1. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let $T : C \to C$ be a contractive mapping, i.e., there exists a real number r with $0 \le r < 1$ such that*

$$\|Tx - Ty\| \le r\|x - y\|, \quad \forall x, y \in C.$$

Then the following hold:

(i) T has a unique fixed point u in C;
(ii) for every $z \in C$, the sequence $\{T^n z\}$ converges to u.

Let H be a real Hilbert space and let C be a nonempty subset of H. Then $U : C \to H$ is called a *contractively strict pseudo-contraction* if there exist $s \in [0, 1)$ and $r \in \mathbb{R}$ with $0 \le r < 1$ such that

$$\|Ux - Uy\|^2 \le s\|x - y\|^2 + r\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C.$$

Using Theorem 3.3, we prove the following unique fixed point theorem.

Theorem 4.2. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let U be a contractively strict pseudo-contraction from C into itself, i.e., there exist $s \in [0, 1)$ and $r \in \mathbb{R}$ with $0 \le r < 1$ such that*

$$\|Ux - Uy\|^2 \le s\|x - y\|^2 + r\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C.$$

Then the following hold:

(i) U has a unique fixed point u in C;
(ii) for every $z \in C$, the sequence $\{(\lambda I + (1 - \lambda)U)^n z\}$ converges to u, where $r \le \lambda < 1$.

Using Theorem 3.1, we have the following theorem for strict pseudo-contractions in a Hilbert space.

Theorem 4.3 ([18]). *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be a strict pseudo-contraction from C into itself, i.e., there exists $r \in \mathbb{R}$ with*

$0 \le r < 1$ *such that*

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + r\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

Let $u \in C$ *and* $s \in (0, 1)$ *with* $r \le s < 1$. *Define a mapping* $U : C \to C$ *as follows:*

$$Ux = su + (1 - s)Tx, \quad \forall x \in C.$$

Then U *has a unique fixed point* z *in* C. *Furthermore, define a mapping* $S : C \to C$ *as follows:*

$$Sx = rx + (1 - r)\big(su + (1 - s)Tx\big), \quad \forall x \in C.$$

Then, for all $x \in C$, *the sequence* $\{S^n x\}$ *converges to a unique fixed point* z.

Using Theorem 3.4, we can prove the following fixed point theorems.

Theorem 4.4. *Let* H *be a real Hilbert space, let* C *be a nonempty bounded closed convex subset of* H *and let* $T : C \to C$ *be contractively nonspreading, i.e., there exists a real number* s *with* $0 \le s < \frac{1}{2}$ *such that*

$$\|Tx - Ty\|^2 \le s\{\|Tx - y\|^2 + \|Ty - x\|^2\}, \quad \forall x, y \in C.$$

Then the following hold:

(i) T *has a unique fixed point* u *in* C;
(ii) *for every* $z \in C$, *the sequence* $\{T^n z\}$ *converges to* u.

Theorem 4.5. *Let* H *be a real Hilbert space, let* C *be a nonempty bounded closed convex subset of* H *and let* $T : C \to C$ *be contractively hybrid, i.e., there exists a real number* s *with* $0 \le s < \frac{1}{3}$ *such that*

$$\|Tx - Ty\|^2 \le s\{\|Tx - y\|^2 + \|Ty - x\|^2 + \|x - y\|^2\}, \quad \forall x, y \in C.$$

Then the following hold:

(i) T *has a unique fixed point* u *in* C;
(ii) *for every* $z \in C$, *the sequence* $\{T^n z\}$ *converges to* u.

Using Theorem 3.4, we obtain an extension of Theorem 1.1.

Theorem 4.6 ([18]). *Let* H *be a real Hilbert space, let* C *be a nonempty bounded closed convex subset of* H *and let* T *be a strict pseudo-contraction from* C *into itself, i.e., there exists* $r \in \mathbb{R}$ *with* $0 \le r < 1$ *such that*

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + r\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

Let $u \in C$ *and* $s_n \in (0, 1)$ *for all* $n \in \mathbb{N}$. *Define a mapping* $U_n : C \to C$ *as follows:*

$$U_n x = s_n u + (1 - s_n)Tx, \quad \forall x \in C, \quad n \in \mathbb{N}.$$

Then the following hold:

(i) U_n *has a unique fixed point* z_n *in* C;
(ii) *if* $s_n \to 0$, *then the sequence* $\{z_n\}$ *converges to* $P_{F(T)}u$, *where* $P_{F(T)}$ *is the metric projection of* H *onto* $F(T)$.

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